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Nonexistence Results of Entire Solutions for Superlinear Elliptic Inequalities

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1. INTRODUCTION

We consider second order elliptic differential inequalities of the form

$$\Delta u \geqslant p(x)\, f(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geqslant 2$, Δ denotes the N-dimensional Laplacian, $p(x)$ is a positive continuous function in $\mathbb{R}^N$, and $f(u)$ is a positive continuous function which is defined either in $\mathbb{R}_+ = (0, \infty)$ or $\mathbb{R}$, and is superlinear at $u = \infty$ in the sense to be specified below. One of the important problems for (1.1) which has been investigated for the past four decades by numerous authors is to establish criteria for the nonexistence of *entire solutions* of (1.1), that is, those C^2-functions $u\colon \mathbb{R}^N \to \operatorname{dom} f$ satisfying (1.1) at every point of $\mathbb{R}^N$; see, e.g., the papers [1, 3, 5–9, 11, 14].

Our objective here is to develop new nonexistence criteria for (1.1) which extend some of the previous ones. In view of the fact that the case $p(x) \equiv \text{const}$ has been studied in full detail, we intend to obtain criteria so as to apply to the case in which $p(x)$ is allowed to decay to zero as $|x| \to \infty$.

The main results of this paper are presented in Sections 2 and 3. In Section 2, inequality (1.1) is considered under the condition that f is convex and nonexistence criteria are derived through the analysis of an ordinary differential inequality satisfied by the spherical mean of a possible entire solution of (1.1). Section 3 concerns inequality (1.1) without the convexity of f and provides nonexistence criteria with the use of comparison method based on the maximum principle.

It would be of interest to generalize the results of Sections 2 and 3 to the inequality

$$Lu \geqslant p(x)\, f(u), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

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where L denotes the elliptic operator

$$L = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial}{\partial x_i}.$$

In Section 4 we show that the method of Section 3 can be applied to yield nonexistence criteria for (1.2) provided $f(u)$ is restricted to $f(u) = u^\sigma$, $\sigma > 1$, or $f(u) = e^u$.

2. The Method of Averaging

We consider inequality (1.1),

$$\Delta u \geqslant p(x) f(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $p(x)$ is positive and continuous in $\mathbb{R}^N$, and $f(u)$ satisfies the following conditions:

(A_1) $f: I \to \mathbb{R}_+$ is convex, where $I = \mathbb{R}_+$ or $\mathbb{R}$. When $I = \mathbb{R}_+$, the limit $\lim_{u \to +0} f(u)$ exists and is finite.

(A_2) $\int_1^\infty [F(z)]^{-1/2}\, dz < \infty$, where

$$F(z) = \int_0^z f(s)\, ds, \qquad z \in I. \tag{2.1}$$

Important special cases of such an f are, for example, u^σ, $\sigma > 1$, e^u, and $u[\log(1+u)]^\delta$, $\delta > 2$.

We now define a continuous function $G: I \to \mathbb{R}_+$ by

$$G(\alpha) = \int_\alpha^\infty [F(z) - F(\alpha)]^{-1/2}\, dz, \qquad \alpha \in I. \tag{2.2}$$

The existence of G is guaranteed by our assumption (A_2). This function G will be extensively used in the sequel.

Lemma 2.1. (i) *G is eventually nonincreasing.*

(ii) $\lim_{\alpha \to \infty} G(\alpha) = 0$.

Proof. Since the convexity of $f(u)$ and the integral condition (A_2) imply that $f(u)$ is eventually nondecreasing and $\lim_{u \to \infty} f(u) = \infty$, the validity of (i) follows from the fact that $G(\alpha)$ can be rewritten as

$$G(\alpha) = \int_0^\infty \left(\int_0^u f(s+\alpha)\, ds \right)^{-1/2} du. \tag{2.3}$$

Furthermore from the monotone convergence theorem it follows that

$$\lim_{\alpha\to\infty} G(\alpha)=\int_0^\infty\left(\int_0^u \lim_{\alpha\to\infty} f(s+\alpha)\,ds\right)^{-1/2} du=0,$$

which proves (ii).

A simple computation shows that if $f(u)=u^\sigma$, $\sigma>1$, then

$$G(\alpha)=\left(\frac{\pi}{\sigma+1}\right)^{1/2}\cdot\frac{\Gamma(\sigma/2(1+\sigma))}{\Gamma(\sigma/1+\sigma)}\,\alpha^{-(\sigma-1)/2},$$

and that if $f(u)=e^u$, then

$$G(\alpha)=\pi e^{-\alpha/2}.$$

Our first nonexistence result concerns the case $N\geqslant 3$ and generalizes [1, Theorem 2.1].

THEOREM 2.1. *Let $N\geqslant 3$ and suppose that (A_1) and (A_2) hold. Suppose moreover that there exist a constant $m>1$ and a nonincreasing continuous function $p_*\colon [0,\infty)\to\mathbb{R}_+$ such that*

$$p(x)\geqslant p_*(|x|),\qquad x\in\mathbb{R}^N, \tag{2.4}$$

$$\int_0^\infty s p_*(s)\,ds=\infty, \tag{2.5}$$

and

$$\liminf_{r\to\infty} G(cP_*(r))\left(\int_r^{mr}[p_*(s)]^{1/2}\,ds\right)^{-1}=0 \qquad \text{for all}\quad c>0, \tag{2.6}$$

where $|\cdot|$ denotes the Euclidean length and

$$P_*(r)=\int_0^r s\left[1-\left(\frac{s}{r}\right)^{N-2}\right]p_*(s)\,ds,\qquad r\geqslant 0. \tag{2.7}$$

(i) *If* $\operatorname{dom} f=\mathbb{R}_+$, *then inequality* (1.1) *has no positive entire solutions.*

(ii) *If* $\operatorname{dom} f=\mathbb{R}$, *then inequality* (1.1) *has no entire solutions.*

Proof. We present the proof of (i) only, because (ii) can be treated similarly. Suppose to the contrary that inequality (1.1) with $\operatorname{dom} f=\mathbb{R}_+$

admits a positive entire solution $u(x)$. Let $\bar{u}(r)$ denote the mean value of $u(x)$ over the sphere $|x| = r$, that is,

$$\bar{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x| = r} u(x)\, dS, \qquad r \geqslant 0,$$

where ω_N is the area of the unit sphere in $\mathbb{R}^N$. Then by taking the mean value of inequality (1.1) over $|x| = r$, and using (2.4) and Jensen's inequality, we see that $\bar{u}(r)$ satisfies

$$(r^{N-1} \bar{u}'(r))' \geqslant r^{N-1} p_*(r)\, f(\bar{u}(r)), \qquad r \geqslant 0. \tag{2.8}$$

Since $\bar{u}'(0) = 0$, the above inequality implies that $\bar{u}'(r) \geqslant 0$, $r \geqslant 0$. Integrating (2.8) twice over $[0, r]$, we have

$$\bar{u}(r) \geqslant \bar{u}(0) + \frac{1}{N-2} \int_0^r s \left[1 - \left(\frac{s}{r}\right)^{N-2} \right] p_*(s)\, f(\bar{u}(s))\, ds, \qquad r \geqslant 0.$$

Since $\inf_{s \geqslant 0} f(\bar{u}(s)) > 0$, there is a constant $C_1 > 0$ such that

$$\bar{u}(r) \geqslant C_1 P_*(r), \qquad r \geqslant 0,$$

where $P_*(r)$ is defined by (2.7). From this inequality and (2.5) it follows that $\bar{u}(r)$ tends monotonically to infinity as $r \to \infty$, so that there is an $R_0 > 0$ such that $f(u)$ is nondecreasing in $[C_1 P_*(R_0), \infty)$.

Integrating inequality (2.8) twice over $[R, r]$, $R \geqslant R_0$, we see that

$$\begin{aligned} \bar{u}(r) &\geqslant \bar{u}(R) + \frac{1}{N-2} \int_R^r s \left[1 - \left(\frac{s}{r}\right)^{N-2} \right] p_*(s)\, f(\bar{u}(s))\, ds \\ &\geqslant C_1 P_*(R) + \frac{1}{N-2} \int_R^r s \left[1 - \left(\frac{s}{r}\right)^{N-2} \right] p_*(s)\, f(\bar{u}(s))\, ds, \quad r \geqslant R \geqslant R_0. \end{aligned} \tag{2.9}$$

Using the simple inequality

$$s \left[1 - \left(\frac{s}{r}\right)^{N-2} \right] \geqslant \frac{N-2}{m^{N-2}} (r - s) \qquad \text{for} \quad R \leqslant s \leqslant r \leqslant mR,$$

in (2.9), we obtain

$$\bar{u}(r) \geqslant C_1 P_*(R) + C_2 \int_R^r (r - s)\, p_*(s)\, f(\bar{u}(s))\, ds, \qquad R \leqslant r \leqslant mR, \tag{2.10}$$

for some constant $C_2 = C_2(m, N) > 0$ independent of R. Now let us define $v(r)$, $R \leqslant r \leqslant mR$, by the right hand side of (2.10). Then $v(r)$ satisfies

$$v(R) = C_1 P_*(R),$$

$$v'(r) = C_2 \int_R^r p_*(s) f(\bar{u}(s))\, ds \geqslant 0$$

and

$$v''(r) = C_2 p_*(r) f(\bar{u}(r)) \geqslant C_2 p_*(r) f(v(r)) \tag{2.11}$$

for $R \leqslant r \leqslant mR$. Multiplying (2.11) by $v'(r) \geqslant 0$ and integrating the resulting inequality on $[R, r]$, we see that

$$[v'(r)]^2 \geqslant 2C_2 p_*(r)[F(v(r)) - F(C_1 P_*(r))]$$

for $R \leqslant r \leqslant mR$, which implies

$$v'(r)[F(v(r)) - F(C_1 P_*(r))]^{-1/2} \geqslant C_3[p_*(r)]^{1/2} \tag{2.12}$$

for $R \leqslant r \leqslant mR$, where $C_3 = (2C_2)^{1/2} > 0$. Finally, an integration of (2.12) over $[R, mR]$ yields

$$\begin{aligned} G(C_1 P_*(R)) &\geqslant \int_{C_1 P_*(R)}^{v(mR)} [F(z) - F(C_1 P_*(R))]^{-1/2}\, dz \\ &\geqslant C_3 \int_R^{mR} [p_*(s)]^{1/2}\, ds, \qquad R \geqslant R_0, \end{aligned}$$

which contradicts (2.6). This completes the proof.

COROLLARY 2.1. *Let $N \geqslant 3$. In addition to* (A_1) *and* (A_2) *suppose that*

$$\liminf_{|x| \to \infty} |x|^2 p(x) > 0. \tag{2.13}$$

(i) *If* $\operatorname{dom} f = \mathbb{R}_+$, *then inequality* (1.1) *has no positive entire solutions.*

(ii) *If* $\operatorname{dom} f = \mathbb{R}$, *then inequality* (1.1) *has no entire solutions.*

Proof. Put

$$p_*(r) = C(r+1)^{-2}, \qquad r \geqslant 0,$$

where $C>0$ is a constant. Because of (2.13), $C>0$ can be chosen so that p_* satisfies (2.4). Clearly p_* satisfies (2.5). Since

$$\int_r^{mr} [p_*(s)]^{1/2}\, ds \geqslant C^{1/2} \log m > 0 \qquad \text{for} \quad m>1, r \geqslant 1,$$

condition (2.6) is satisfied. The conclusion then follows immediately from Theorem 2.1.

The next example shows that the condition (2.13) on $p(x)$ is in some cases critical.

EXAMPLE 2.1. Let $N \geqslant 3$. Consider the equation

$$\Delta u = c(x)\, f(u), \qquad x \in \mathbb{R}^N, \tag{2.14}$$

where c is positive and continuous in $\mathbb{R}^N$ and f satisfies conditions (A_1) and (A_2) with $\operatorname{dom} f = \mathbb{R}_+$. Theorem 2.1 implies that if

$$\liminf_{|x| \to \infty} |x|^2\, c(x) > 0,$$

then Eq. (2.14) has no positive entire solutions. On the other hand, it is easily verified that if $f(+0)=0$ and $c(x)$ is locally Hölder continuous in $\mathbb{R}^N$ and satisfies

$$\limsup_{|x| \to \infty} |x|^{2+\varepsilon}\, c(x) < \infty$$

for some $\varepsilon > 0$, then Eq. (2.14) has a positive entire solution which tends to a positive constant as $|x| \to \infty$ (see, for example, [7]).

A nonexistence result for the case $N=2$ now follows.

THEOREM 2.2. *Let $N=2$ and suppose that* (A_1) *and* (A_2) *hold. Suppose moreover that there exists a continuous function* $p_*\colon [0, \infty) \to \mathbb{R}_+$ *and* $r_0 > 0$ *satisfying*

$$p(x) \geqslant p_*(|x|), \qquad x \in \mathbb{R}^2, \tag{2.15}$$

$$r^2 p_*(r) \text{ is nonincreasing for } r \geqslant r_0, \tag{2.16}$$

and either

$$\int_0^{\infty} [p_*(s)]^{1/2}\, ds = \infty, \tag{2.17}$$

or

$$\liminf_{r\to\infty} G(c\bar{P}_*(r))\left(\int_r^\infty [p_*(s)]^{1/2}\,ds\right)^{-1} < \sqrt{2} \qquad \textit{for all} \quad c>0, \tag{2.18}$$

where

$$\bar{P}_*(r) = \int_0^r s\log\left(\frac{r}{s}\right)\cdot p_*(s)\,ds, \qquad r\geqslant 0.$$

(i) *If* $\operatorname{dom} f = \mathbb{R}_+$, *then inequality* (1.1) *has no positive entire solutions.*

(ii) *If* $\operatorname{dom} f = \mathbb{R}$, *then inequality* (1.1) *has no entire solutions.*

Proof. We consider the case $\operatorname{dom} f = \mathbb{R}_+$. Suppose to the contrary that there is a positive entire solution $u(x)$ of (1.1). Let $\bar{u}(r)$ be the mean value of $u(x)$ over the circle $|x| = r$, $r\geqslant 0$. Then $\bar{u}(r)$ satisfies inequality (2.8) (with $N=2$) as well as the initial condition $\bar{u}'(0)=0$, and as in the proof of Theorem 2.1, it can be shown that

$$\bar{u}(r)\geqslant C\bar{P}_*(r), \qquad r\geqslant 0, \tag{2.19}$$

for some $C>0$. We now multiply (2.8) ($N=2$) by $2r\bar{u}'(r)\geqslant 0$, rewrite the resulting inequality as

$$[(r\bar{u}'(r))^2]'\geqslant 2r^2p_*(r)[F(\bar{u}(r))]', \qquad r\geqslant 0,$$

and integrate the above over $[R, r]$, $R\geqslant r_0$. Then, using (2.16), we have

$$(r\bar{u}'(r))^2-(R\bar{u}'(R))^2\geqslant 2r^2p_*(r)[F(\bar{u}(r))-F(\bar{u}(R))] \tag{2.20}$$

for $r\geqslant R\geqslant r_0$, whence it follows that

$$\bar{u}'(r)[F(\bar{u}(r))-F(\bar{u}(R))]^{-1/2}\geqslant\sqrt{2}\,[p_*(r)]^{1/2}$$

for $r\geqslant R\geqslant r_0$. An integration of this inequality over $[R,\infty)$ yields

$$G(\bar{u}(R))\geqslant\sqrt{2}\int_R^\infty [p_*(s)]^{1/2}\,ds, \qquad R\geqslant r_0. \tag{2.21}$$

This implies $\int_0^\infty [p_*(s)]^{1/2}\,ds<\infty$, which contradicts (2.17). Since G is eventually nonincreasing by (i) of Lemma 2.1, from (2.19) and (2.21) we see that

$$G(C\bar{P}_*(R))\geqslant\sqrt{2}\int_R^\infty [p_*(s)]^{1/2}\,ds$$

for sufficiently large R, which contradicts (2.18). This completes the proof of the first statement. The second statement can be proved similarly.

The following theorem shows that the nonexistence criterion of Theorem 2.2 can be somewhat strengthened if the function $f(u)$ is required to satisfy the superlinear condition

(A_2') $\int_1^\infty [F(z)]^{-\lambda/2}\, ds < \infty$ for some $\lambda \in (0, 1)$,

which is stronger than (A_2). In this case we use, instead of G, the function G_λ defined by

$$G_\lambda(\alpha) = \int_\alpha^\infty [F(z) - F(\alpha)]^{-\lambda/2}\, dz, \qquad \alpha \in I.$$

THEOREM 2.3. *Let $N = 2$ and suppose that (A_1) and (A_2') hold. Suppose moreover that there exist a continuous function $p_*: [0, \infty) \to \mathbb{R}_+$ and $r_0 > 0$ satisfying*

$$p(x) \geqslant p_*(|x|), \qquad x \in \mathbb{R}^2, \tag{2.22}$$

$$r^2 p_*(r) \text{ is nonincreasing for } r \geqslant r_0, \tag{2.23}$$

and either

$$\int_1^\infty s^{-(1-\lambda)} [p_*(s)]^{\lambda/2}\, ds = \infty, \tag{2.24}$$

or

$$\liminf_{r \to \infty} G_\lambda(c\bar{P}_*(r)) \left(\int_0^r s p_*(s) f(c\bar{P}_*(s))\, ds \right)^{-(1-\lambda)} \cdot \left(\int_r^\infty s^{-(1-\lambda)} [p_*(s)]^{\lambda/2}\, ds \right)^{-1} = 0 \qquad \text{for all } c > 0. \tag{2.25}$$

(i) *If* $\operatorname{dom} f = \mathbb{R}_+$, *then inequality* (1.1) *has no positive entire solutions.*

(ii) *If* $\operatorname{dom} f = \mathbb{R}$, *then inequality* (1.1) *has no entire solutions.*

Remark 2.1. Under the condition (2.23) for some $r_0 > 0$, (2.17) implies (2.24).

Proof of Theorem 2.3. Suppose to the contrary that there is an entire solution $u: \mathbb{R}^N \to \operatorname{dom} f$ of (1.1). The mean value $\bar{u}(r)$ of $u(x)$ over the circle $|x| = r$, $r \geqslant 0$, satisfies (2.8) (with $N = 2$) and (2.19). In particular,

$\bar{u}(r) \to \infty$ as $r \to \infty$, so that there exists an $R_0 \geqslant r_0$ such that (2.20) holds for $r \geqslant R \geqslant R_0$. It is easy to derive the following inequality from (2.20):

$$(\bar{u}'(r))^2 \geqslant C_1 (R\bar{u}'(R))^{2(1-\lambda)} r^{-2(1-\lambda)} [p_*(r)]^\lambda [F(\bar{u}(r)) - F(\bar{u}(R))]^\lambda \tag{2.26}$$

for $r \geqslant R \geqslant R_0$, where $C_1 = C_1(\lambda) > 0$ is some constant. On the other hand, integrating (2.8) ($N = 2$) and using (2.19), we find

$$R\bar{u}'(R) \geqslant \int_{R_1}^{R} sp_*(s) f(\bar{u}(s))\, ds \geqslant \int_{R_1}^{R} sp_*(s) f(C\bar{P}_*(s))\, ds$$

for $R \geqslant R_1$, where $R_1 \geqslant R_0$ is sufficiently large. Combining this fact with (2.26), we have

$$\begin{aligned} &\bar{u}'(r)[F(\bar{u}(r)) - F(\bar{u}(R))]^{-\lambda/2} \\ &\quad \geqslant C_2 \left(\int_{R_1}^{R} sp_*(s) f(C\bar{P}_*(s))\, ds \right)^{1-\lambda} r^{-(1-\lambda)} [p_*(r)]^{\lambda/2} \end{aligned} \tag{2.27}$$

for $r \geqslant R \geqslant R_1$ and some constant $C_2 > 0$. An integration of (2.27) over $[R, \infty)$ then gives

$$G_\lambda(\bar{u}(R)) \geqslant C_2 \left(\int_{R_1}^{R} sp_*(s) f(C\bar{P}_*(s))\, ds \right)^{1-\lambda} \left(\int_{R}^{\infty} s^{-(1-\lambda)} [p_*(s)]^{\lambda/2}\, ds \right) \tag{2.28}$$

for $R \geqslant R_1$, which implies in particular that $\int_1^\infty s^{-(1-\lambda)} [p_*(s)]^{\lambda/2}\, ds < \infty$. This contradicts (2.24). Since G_λ is shown to be eventually nonincreasing, as in the proof of Lemma 2.1, from (2.28) and (2.19) we have

$$G_\lambda(C\bar{P}_*(R)) \geqslant C_2 \left(\int_{R_1}^{R} sp_*(s) f(C\bar{P}_*(s))\, ds \right)^{1-\lambda} \left(\int_{R}^{\infty} s^{-(1-\lambda)} [p_*(s)]^{\lambda/2}\, ds \right)$$

for large R. But this contradicts (2.25). This completes the proof.

EXAMPLE 2.2. Let $N = 2$ and suppose that (A_1) and (A_2') hold. If

$$\liminf_{|x| \to \infty} |x|^2 (\log |x|)^\alpha\, p(x) > 0 \qquad \text{for some} \quad \alpha \in [0, 2/\lambda] \tag{2.29}$$

then, as is easily verified, there is a continuous function $p_*\colon [0, \infty) \to \mathbb{R}_+$ such that (2.22), (2.23), and (2.24) are satisfied, so that Theorem 2.3 implies that (1.1) has no entire solutions $u\colon \mathbb{R}^N \to \operatorname{dom} f$. Since (A_2') implies

(A_2), one can also apply Theorem 2.2 to conclude that the nonexistence of entire solutions $u: \mathbb{R}^N \to \operatorname{dom} f$ of (1.1) is guaranteed if

$$\liminf_{|x| \to \infty} |x|^2 (\log |x|)^\alpha p(x) > 0 \qquad \text{for some} \quad \alpha \in [0, 2]. \tag{2.30}$$

Obviously, (2.29) is better that (2.30).

Remark 2.2. We can reduce (2.25) to the weaker form

$$\liminf_{r \to \infty} \left(\int_1^r s p_*(s) f(c \log s)\, ds \right)^{1-\lambda} \cdot \left(\int_r^\infty s^{-(1-\lambda)} [p_*(s)]^{\lambda/2}\, ds \right) > 0 \qquad \text{for all} \quad c > 0.$$

This follows, from the fact that $\lim_{\alpha \to \infty} G_\lambda(\alpha) = 0$, which can be easily proved, and

$$\liminf_{r \to \infty} P_*(r)/\log r > 0.$$

Obviously, this condition automatically implies

$$\int_1^\infty r (\max_{|x|=r} p(x)) f(c \log r)\, dr = \infty \qquad \text{for all} \quad c > 0.$$

Conversely, if

$$\int_1^\infty r (\max_{|x|=r} p(x)) f(c \log r)\, dr < \infty \qquad \text{for some} \quad c > 0,$$

and suitable additional conditions on f hold, then inequality (1.1) admits entire solutions having logarithmic growth at infinity (see [1]).

3. The Comparison Method

The purpose of this section is to show that criteria for the nonexistence of entire solutions of (1.1) can also be obtained by means of a comparison method based on the maximum principle.

Throughout this section we assume that $p(x)$ is positive and continuous in $\mathbb{R}^N$ and $f(u)$ satisfies the following conditions.

(B_1) $f: I \to \mathbb{R}_+$ is locally Lipschitzian and strictly increasing, where $I = \mathbb{R}_+$ or $\mathbb{R}$.

(B_2) $\int_1^\infty [F(z)]^{-1/2}\, dz < \infty$, $F(z) = \int_0^z f(s)\, ds$.

(B_3) In case $I = \mathbb{R}_+$, we assume additionally that

$$\int_{0+}^{1} [F(z)]^{-1/2}\, dz = \infty.$$

Under the above conditions the function G defined by (2.2) is strictly decreasing on $I = \operatorname{dom} f$ and satisfies $\lim_{\alpha \to 0+} G(\alpha) = \infty$ if $\operatorname{dom} f = \mathbb{R}_+$ and $\lim_{\alpha \to -\infty} G(\alpha) = \infty$ if $\operatorname{dom} f = \mathbb{R}$. Clearly, the inverse function G^{-1} of G exists and is strictly decreasing on $\mathbb{R}_+$. We use the notation $H = G^{-1}$, in terms of which the nonexistence theorems to follow are formulated.

First, we prove a lemma on which the proofs of our nonexistence theorems are based.

LEMMA 3.1. *Let k and R be positive constants. If u is a C^2-function satisfying the inequality*

$$\Delta u \geqslant k f(u) \qquad \text{in} \quad |x - x^0| \leqslant R,$$

then

$$u(x^0) \leqslant H\left(\left[\frac{2k}{N}\right]^{1/2} R\right). \tag{3.1}$$

Proof. It is not hard to construct a C^2-function w which depends only on $|x - x^0|$ and satisfies

$$\Delta w = k f(w) \text{ in } |x - x^0| < R, \qquad w(x) \to \infty \text{ as } |x - x^0| \to R. \tag{3.2}$$

(See, e.g., [5, 13]; note that condition (B_3) is essentially used here in case $\operatorname{dom} f = \mathbb{R}_+$.) It can be shown that

$$u(x) \leqslant w(x) \qquad \text{in} \quad |x - x^0| < R. \tag{3.3}$$

In fact, if (3.3) fails to hold, then, since $u(x) - w(x) \to -\infty$ as $|x - x^0| \to R$, there exists a point x^*, $|x^* - x^0| < R$, at which $u(x) - w(x)$ takes the positive (absolute) maximum in $|x - x^0| < R$. Since $f(u)$ is strictly increasing, we then have

$$0 \geqslant \Delta(u - w)(x^*) \geqslant k[f(u(x^*)) - f(w(x^*))] > 0,$$

which is a contradiction. Thus we must have (3.3), which implies in particular,

$$u(x^0) \leqslant w(x^0). \tag{3.4}$$

Let ε be an arbitrary constant such that $0<\varepsilon<\min\{1, R\}$ and let $v(r)$ be a C^2-function which satisfies

$$\begin{aligned} &v''(r)=((1-\varepsilon)/N)\,kf(v(r)) \qquad \text{for} \quad 0\leqslant r<R-\varepsilon, \\ &v'(0)=0 \qquad \text{and} \qquad v(r)\to\infty \ \text{ as } \ r\to R-\varepsilon. \end{aligned} \tag{3.5}$$

The existence of v is a consequence of a simple quadrature (and condition (B_3) if $\operatorname{dom} f=\mathbb{R}_+$). We claim that

$$w(x^0)<v(0). \tag{3.6}$$

If $w(x^0)\geqslant v(0)$, then applying a result of Walter and Rhee [15, Lemma 1] to (3.2) and (3.5), we have

$$w(x)\geqslant v(|x-x^0|) \qquad \text{for} \quad 0<|x-x^0|<R-\varepsilon,$$

which implies $w(x)\to\infty$ as $|x-x^0|\to R-\varepsilon$. But this is a contradiction since w is defined for $|x-x^0|<R$, and proves the truth of (3.6). From (3.4) and (3.6) we see that

$$u(x^0)<v(0).$$

Combining this inequality with the fact that

$$G(v(0))\equiv\int_{v(0)}^{\infty}[F(z)-F(v(0))]^{-1/2}\,dz=\left[\frac{2k(1-\varepsilon)}{N}\right]^{1/2}(R-\varepsilon),$$

which is easy to verify, we conclude that

$$u(x^0)<H\left(\left[\frac{2k(1-\varepsilon)}{N}\right]^{1/2}(R-\varepsilon)\right).$$

Letting $\varepsilon\to 0$ in the above, we obtain (3.1) as desired. This completes the proof.

Our main results below are formulated in terms of the functions p_*, $m\in C[0,\infty)$ saisfying

$$\min_{|x|=r} p(x)\geqslant p_*(r)>0, \qquad r\geqslant 0; \tag{3.7}$$

$$\min_{r/2\leqslant|x|\leqslant 3r/2} p(x)\geqslant m(r)>0, \qquad r\geqslant 0. \tag{3.8}$$

THEOREM 3.1. *Let $N\geqslant 3$. Suppose that* (B_1) *and* (B_2) *hold.*

(i) *Let* dom $f=\mathbb{R}_+$ *and suppose moreover that* (B_3) *holds. If*

$$\liminf_{r\to\infty} H\left(r\left[\frac{m(r)}{2N}\right]^{1/2}\right)\left(\int_0^r s\left[1-\left(\frac{s}{r}\right)^{N-2}\right]p_*(s)\,ds\right)^{-1}=0, \tag{3.9}$$

then inequality (1.1) *has no positive entire solutions.*

(ii) *Let* dom $f=\mathbb{R}$. *If*

$$\int_0^\infty sp_*(s)\,ds=\infty \tag{3.10}$$

and

$$\liminf_{r\to\infty} H\left(r\left[\frac{m(r)}{2N}\right]^{1/2}\right)\left(\int_0^r s\left[1-\left(\frac{s}{r}\right)^{N-2}\right]p_*(s)\,ds\right)^{-1}\leqslant 0, \tag{3.11}$$

then inequality (1.1) *has no entire solutions.*

THEOREM 3.2. *Let* $N=2$. *Suppose that* (B_1) *and* (B_2) *hold.*

(i) *Let* dom $f=\mathbb{R}_+$ *and suppose moreover that* (B_3) *holds. If*

$$\liminf_{r\to\infty} H\left(\frac{r\sqrt{m(r)}}{2}\right)\left(\int_0^r s\log\left(\frac{r}{s}\right)\cdot p_*(s)\,ds\right)^{-1}=0,$$

then inequality (1.1) *has no positive entire solutions.*

(ii) *Let* dom $f=\mathbb{R}$. *If*

$$\liminf_{r\to\infty} H\left(\frac{r\sqrt{m(r)}}{2}\right)\left(\int_0^r s\log\left(\frac{r}{s}\right)\cdot p_*(s)\,ds\right)^{-1}\leqslant 0,$$

then inequality (1.1) *has no entire solutions.*

Proof of Theorem 3.1. Suppose to the contrary that we have an entire solution $u\colon\mathbb{R}^N\to\operatorname{dom} f$. First we show that the following estimate holds:

$$u(x)\leqslant H\left(|x|\left[\frac{m(|x|)}{2N}\right]^{1/2}\right),\qquad x\neq 0. \tag{3.12}$$

Let $x^0\neq 0$ be fixed arbitrarily and put $r=|x^0|>0$. By the definition of $m(r)$ and inequality (1.1), u satisfies

$$\Delta u\geqslant m(r)\,f(u),\qquad |x-x^0|\leqslant r/2.$$

Therefore, Lemma 3.1 asserts that

$$u(x^0) \leqslant H\left(\frac{r}{2}\left[\frac{2m(r)}{N}\right]^{1/2}\right),$$

which proves (3.12).

Now, we adapt the device employed by Redheffer [12] to obtain Liouville type theorems. Since u is not identically any constant, the function $U(r) \equiv \max_{|x|=r} u(x)$, $r \geqslant 0$, is strictly increasing. Put

$$v(r) = \frac{1}{2(N-2)} \int_0^r s\left[1 - \left(\frac{s}{r}\right)^{N-2}\right] p_*(s)\, ds, \qquad r \geqslant 0. \tag{3.13}$$

Then it is easy to see that

$$v(r) > 0, \qquad v'(r) > 0, \qquad r > 0,$$

and

$$\Delta v(|x|) = \tfrac{1}{2} p_*(|x|) < p(x), \qquad x \in \mathbb{R}^N.$$

Choose $r_0 > 0$ arbitrarily and fix it.

(i) Let dom $f = \mathbb{R}_+$. We note that

$$\liminf_{|x| \to \infty} \frac{u(x)}{v(|x|)} = 0 \tag{3.14}$$

by (3.9) and (3.12). Since $u(x) > 0$ on $|x| = r_0$ and $\partial u/\partial v > 0$ for some x, $|x| = r_0$, where v denotes the outward normal vector to the sphere, there is a sufficiently small $\delta > 0$ such that

$$u(x) - \delta v(|x|) > 0 \qquad \text{on} \quad |x| = r_0$$

and

$$\frac{\partial}{\partial v}(u(x) - \delta v(|x|)) > 0 \qquad \text{for some } x,\ |x| = r_0.$$

On the other hand, in view of (3.14) we see that

$$\liminf_{|x| \to \infty} (u(x) - \delta v(|x|)) = \liminf_{|x| \to \infty} v(|x|)\left(\frac{u(x)}{v(|x|)} - \delta\right) < 0.$$

It follows that $u(x) - \delta v(|x|)$ takes a local positive maximum at an interior point x^* of the region $|x| > r_0$. We may suppose moreover that

$u(x^*) \geqslant U(r_0)$. Let us now choose $\delta > 0$ so that $\delta \leqslant f(U(r_0))$, which is possible since δ can be made as small as possible. Then we have

$$\begin{aligned} \Delta u(x^*) &\leqslant \delta\, \Delta v(|x^*|) < \delta p(x^*) \\ &\leqslant p(x^*)\, f(U(r_0)) \leqslant p(x^*)\, f(u(x^*)), \end{aligned}$$

which contradicts the assumption that u is a solution of (1.1).

(ii) Let $\operatorname{dom} f = \mathbb{R}$. Note that (3.10) implies that $v(r) \to \infty$ and that

$$\liminf_{|x| \to \infty} \frac{u(x)}{v(|x|)} \leqslant 0$$

by (3.11) and (3.12). Then, since

$$\liminf_{|x| \to \infty} (u(x) - \delta v(|x|)) = \liminf_{|x| \to \infty} v(|x|) \left(\frac{u(x)}{v(|x|)} - \delta \right) = -\infty$$

for any $\delta > 0$, we can find a $\delta > 0$ sufficiently small such that the function $u(x) - \delta v(|x|)$ attains a local maximum at an interior point of $|x| > r_0$. From this point on we proceed exactly as in the proof of (i) to reach the desired contradiction. The proof of Theorem 3.1 is complete.

To prove Theorem 3.2 it suffices to repeat essentially the same argument as above by replacing the function (3.13) by

$$v(r) = \frac{1}{2} \int_0^r s \log\left(\frac{r}{s}\right) \cdot p_*(s)\, ds, \qquad r \geqslant 0.$$

The observation that $v(r) \to \infty$ as $r \to \infty$ will be useful.

The next corollary, which can be regarded as a variant of Corollary 2.1, is a simple consequence of Theorem 3.1 and the decreasing nature of H.

COROLLARY 3.1. *Let $N \geqslant 3$. Suppose that* (B_1) *and* (B_2) *hold, and that*

$$\liminf_{|x| \to \infty} |x|^2 p(x) > 0. \tag{2.13}$$

(i) *If* $\operatorname{dom} f = \mathbb{R}_+$ *and* (B_3) *holds, then inequality* (1.1) *has no positive entire solutions.*

(ii) *If* $\operatorname{dom} f = \mathbb{R}$, *then inequality* (1.1) *has no entire solutions.*

4. Generalizations

It is natural to ask if the results of the preceding sections can be extended to more general inequalities of the form $Lu \geqslant p(x) f(u)$, $x \in \mathbb{R}^N$, where L is the elliptic operator defined by

$$L \equiv \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial}{\partial x_i}.$$

Since the problem in this general setting seems to be difficult, we restrict our attention to the particular cases that $f(u) = u^\sigma$, $\sigma > 1$, and $f(u) = e^u$, that is, to the inequalities

$$Lu \geqslant p(x) u^\sigma, \qquad x \in \mathbb{R}^N, \qquad \sigma > 1, \tag{4.1}$$

and

$$Lu \geqslant p(x) e^u, \qquad x \in \mathbb{R}^N, \tag{4.2}$$

and show that the comparison method employed in Section 3 may be applied to derive criteria for the nonexistence of entire solutions of these inequalities. For related results we refer to the papers [4, 6, 10, 12, 14].

It is assumed that $N \geqslant 2$, a_{ij}, b_i, and p are continuous in $\mathbb{R}^N$, $1 \leqslant i, j \leqslant N$, $p(x) > 0$ in $\mathbb{R}^N$ and the matrix $(a_{ij}(x))$ is symmetric and positive definite for $x \in \mathbb{R}^N$.

Lemma 4.1. *Let p_0 and R be positive constants and let $x^0 = (x_i^0) \in \mathbb{R}^N$.*

(i) *If a positive C^2-function u satisfies the inequality*

$$Lu \geqslant p_0 u^\sigma, \qquad |x - x^0| \leqslant R,$$

then we have

$$u(x^0) \leqslant \left(\frac{4\mu T}{p_0}\right)^{1/(\sigma-1)} R^{-\mu}, \tag{4.3}$$

where $\mu = 2/(\sigma - 1)$, and $T = T(x^0, R) > 0$ is a constant such that

$$T \geqslant \sup_{|x-x^0| \leqslant R} \frac{2(\mu+1)}{|x-x^0|^2} \sum_{i,j=1}^{N} a_{ij}(x)(x_i - x_i^0)(x_j - x_j^0),$$

$$T \geqslant \sup_{|x-x^0| \leqslant R} \sum_{i=1}^{N} (a_{ii}(x) + b_i(x)(x_i - x_i^0)), \qquad x = (x_i).$$

(ii) *If a C^2-function u satisfies the inequality*

$$Lu \geqslant p_0 e^u, \qquad |x-x^0| \leqslant R,$$

then we have

$$u(x^0) \leqslant \log\left(\frac{4\tilde{T}}{p_0 R^2}\right),$$

where $\tilde{T} = \tilde{T}(x^0, R) > 0$ is a constant such that

$$\tilde{T} \geqslant \sup_{|x-x^0| \leqslant R} \frac{2}{|x-x^0|^2} \sum_{i,j=1}^{N} a_{ij}(x)(x_i - x_i^0)(x_j - x_j^0),$$

$$\tilde{T} \geqslant \sup_{|x-x^0| \leqslant R} \sum_{i=1}^{N} (a_{ii}(x) + b_i(x)(x_i - x_i^0)), \qquad x = (x_i).$$

Proof. (i) Put

$$v(x) = k(R^2 - r^2)^{-\mu},$$

where $k = (4\mu T R^2/p_0)^{1/(\sigma-1)}$ and $r = |x - x^0|$. Then it can be shown that

$$Lv \leqslant p_0 v^\sigma,\ |x-x^0| < R, \qquad v \to \infty \text{ as } |x-x^0| \to R.$$

In fact, it is clear that $v \to \infty$ as $r \to R$, and the first inequality follows from the computation

$$\begin{aligned} Lv(x) = (R^2 - r^2)^{-\mu-2} \Bigg[& 4k\mu(\mu+1) \sum_{i,j=1}^{N} a_{ij}(x)(x_i - x_i^0)(x_j - x_j^0) \\ & + 2k\mu(R^2 - r^2) \sum_{i=1}^{N} (a_{ii}(x) + b_i(x)(x_i - x_i^0)) \Bigg] \\ \leqslant {} & 4k\mu T R^2 (R^2 - r^2)^{-\mu-2} = p_0[v(x)]^\sigma, \qquad |x-x^0| < R. \end{aligned}$$

Applying the argument used in the proof of the first part of Lemma 3.1 (with L replacing Δ), we see that

$$u(x) \leqslant v(x), \qquad |x-x^0| < R,$$

which gives (4.3) for $x = x^0$.

(ii) This estimate is proved in [14, Lemma 2].

In order to state our main results let us introduce some notation. Recall that p_*, $m \in C[0, \infty)$ are defined by (3.7) and (3.8), respectively. Moreover for a fixed $r_0>0$, T, $\tilde{T}$, D, $q\in C[r_0,\infty)$ are defined to be the functions which fulfill for $r\geqslant r_0$

$$T(r)\geqslant \sup_{\substack{|x|=r,\\ |x-y|\leqslant r/2}} \frac{2(\mu+1)}{|x-y|^2}\sum_{i,j=1}^{N} a_{ij}(y)(x_i-y_i)(x_j-y_j),$$

$$T(r)\geqslant \sup_{\substack{|x|=r,\\ |x-y|\leqslant r/2}} \sum_{i=1}^{N} (a_{ii}(y)+b_i(y)(y_i-x_i));$$

$$\tilde{T}(r)\geqslant \sup_{\substack{|x|=r,\\ |x-y|\leqslant r/2}} \frac{2}{|x-y|^2}\sum_{i,j=1}^{N} a_{ij}(y)(x_i-y_i)(x_j-y_j),$$

$$\tilde{T}(r)\geqslant \sup_{\substack{|x|=r,\\ |x-y|\leqslant r/2}} \sum_{i=1}^{N} (a_{ii}(y)+b_i(y)(y_i-x_i));$$

$$D(r)\geqslant \max_{|x|=r} \frac{1}{A(x)}\sum_{i=1}^{N} (a_{ii}(x)+x_i b_i(x));$$

$$0<q(r)\leqslant \min_{|x|=r} \frac{p(x)}{A(x)},$$

where $x=(x_i)$, $y=(y_i)$, and $A(x)$ is given by

$$A(x)=|x|^{-2}\sum_{i,j=1}^{N} a_{ij}(x)x_i x_j, \qquad |x|\geqslant r_0.$$

THEOREM 4.1. *If*

$$\liminf_{r\to\infty}\left(\frac{T(r)}{r^2 m(r)}\right)^{1/(\sigma-1)}\left(\int_{r_0}^{r} K(s)\int_{r_0}^{s}\frac{q(\xi)}{K(\xi)}\,d\xi\,ds\right)^{-1}=0, \tag{4.4}$$

where

$$K(r)=r\exp\left(-\int_{r_0}^{r}\frac{D(s)}{s}\,ds\right), \qquad r\geqslant r_0, \tag{4.5}$$

then inequality (4.1) *has no positive entire solutions.*

THEOREM 4.2. *If*

$$\int_{r_0}^{\infty} K(s)\int_{r_0}^{s}\frac{q(\xi)}{K(\xi)}\,d\xi\,ds=\infty$$

and

$$\liminf_{r\to\infty} \log\left(\frac{\tilde{T}(r)}{r^2 m(r)}\right)\left(\int_{r_0}^{r} K(s)\int_{r_0}^{s}\frac{q(\xi)}{K(\xi)}\,d\xi\,ds\right)^{-1} \leqslant 0,$$

where $K(r)$ *is defined by* (4.5), *then inequality* (4.2) *has no entire solutions.*

Proof of Theorem 4.1. Suppose to the contrary that there is a positive entire solution $u(x)$ of inequality (4.1). We first show that the estimate

$$0 < u(x) \leqslant C\left(\frac{T(|x|)}{|x|^2\, m(|x|)}\right)^{1/(\sigma-1)}, \qquad |x| \geqslant r_0, \tag{4.6}$$

holds, where $C = C(\sigma) > 0$ is independent of x. Let x^0, $|x^0| = r \geqslant r_0$, be fixed arbitrarily. Then by (4.1) and the definition of $m(r)$, we have

$$Lu \geqslant m(r)u^{\sigma}, \qquad |x - x^0| \leqslant r/2.$$

Therefore, from Lemma 4.1 (i) and the definition of $T(r)$ it follows that

$$u(x^0) \leqslant \left(\frac{4\mu T(r)}{m(r)}\right)^{1/(\sigma-1)}\left(\frac{r}{2}\right)^{-\mu},$$

which shows the validity of (4.6).

Next we introduce a comparison function $v(r)$, $r \geqslant r_0$, defined by

$$v(r) = \frac{1}{2}\int_{r_0}^{r} K(s)\int_{r_0}^{s}\frac{q(\xi)}{K(\xi)}\,d\xi\,ds, \qquad r \geqslant r_0.$$

Then it is easily verified that

$$Lv(|x|) < p(x), \qquad |x| \geqslant r_0.$$

Actually, using the definitions of $D(r)$ and $q(r)$ and the fact that $v'(r) \geqslant 0$, $r \geqslant r_0$, we obtain

$$\begin{aligned}\frac{1}{A(x)}\,Lv(|x|) &= v''(|x|) + \frac{1}{|x|}\left[\frac{1}{A(x)}\sum_{i=1}^{N}(a_{ii}(x) + x_i b_i(x)) - 1\right] v'(|x|)\\ &\leqslant v''(|x|) + \frac{1}{|x|}\,[D(|x|) - 1]\,v'(|x|)\\ &= \frac{1}{2}\,q(|x|) < \frac{1}{A(x)}\,p(x), \qquad |x| \geqslant r_0.\end{aligned}$$

From the above observation and our hypothesis (4.4) we find

$$\liminf_{|x| \to \infty} \frac{u(x)}{v(|x|)} = 0.$$

Then, proceeding as in the proof of Theorem 3.1, we can prove the existence of a constant $\delta > 0$ such that the function $u(x) - \delta v(|x|)$ takes a maximum at some point x^*, $|x^*| > r_0$, which immediately leads to a contradiction. This completes the proof.

Proof of Theorem 4.2. A crucial step is to show that an entire solution u of (4.2), if any, is subjected to the estimate

$$u(x) \leqslant \log\left(\frac{16\tilde{T}(|x|)}{|x|^2\, m(|x|)}\right), \qquad |x| \geqslant r_0.$$

The second part of Lemma 4.1 is used for this purpose. Since the remainder of the proof proceeds as in the proof of Theorem 4.1, the detailed verification is left to the reader.

EXAMPLE 4.1. Let $\lambda_i(x)$, $1 \leqslant i \leqslant N$, denote the eigenvalues of the matrix $(a_{ij}(x))$. Suppose that there exist constants C_1, $C_2 > 0$ and α, $\beta \in \mathbb{R}$ such that

$$C_1 |x|^\alpha \leqslant \lambda_i(x) \leqslant C_2 |x|^\alpha \qquad \text{for large } |x|,$$

and

$$b_i(x) = O(|x|^\beta) \qquad \text{as} \quad |x| \to \infty, \quad 1 \leqslant i \leqslant N.$$

Suppose moreover that

$$\liminf_{|x| \to \infty} |x|^{2-\nu}\, p(x) > 0$$

for some $\nu \geqslant \max\{\alpha, \beta + 1\}$. Then, inequality (4.1) admits no positive entire solutions. Similarly, under these conditions, inequality (4.2) admits no entire solutions. To see why it suffices to note that we can take for $r \geqslant r_0$

$$T(r) = C_3(r^\alpha + r^{\beta+1}), \qquad D(r) = C_4 + C_5 r^{\beta+1-\alpha}, \quad \text{and} \quad m(r) = C_6 r^{\nu-2}$$

for suitable constants $C_i > 0$, $3 \leqslant i \leqslant 6$.

Now we shall present some corollaries which can be obtained from Theorems 4.1 and 4.2.

COROLLARY 4.1. *Let $N \geqslant 3$. Suppose that the operator L is uniformly elliptic in $\mathbb{R}^N$, and $b_i(x) = O(|x|^{-1})$ as $|x| \to \infty$, $1 \leqslant i \leqslant N$.*

(i) *If*

$$\liminf_{r \to \infty} (r^2 m(r))^{-1/(\sigma-1)} \left(\int_{r_0}^{r} s \left[1 - \left(\frac{s}{r} \right)^{k-2} \right] p_*(s)\, ds \right)^{-1} = 0 \tag{4.7}$$

for some $k > 2$, then inequality (4.1) *has no positive entire solutions.*

(ii) *If*

$$\int_{r_0}^{\infty} s p_*(s)\, ds = \infty$$

and

$$\limsup_{r \to \infty} \log(r^2 m(r)) \left(\int_{r_0}^{r} s \left[1 - \left(\frac{s}{r} \right)^{k-2} \right] p_*(s)\, ds \right)^{-1} \geqslant 0$$

for some $k > 2$, then inequality (4.2) *has no entire solutions.*

Proof. Only (i) is considered. By our assumptions we can take

$$T(r) = C_1, \qquad D(r) = n, \qquad q(r) = C_2 p_*(r)$$

for some appropriate constants $n > 2$, C_1, $C_2 > 0$, and hence we have

$$K(r) = C_3 r^{1-n}$$

for some constant $C_3 > 0$. On the other hand, when $n > k$, the inequality

$$1 - \left(\frac{s}{r} \right)^{n-2} \geqslant 1 - \left(\frac{s}{r} \right)^{k-2}, \qquad r_0 \leqslant s \leqslant r,$$

holds. From these observations it follows that (4.4) is implied by (4.7). This completes the proof.

Our final result concerns the case that L is a perturbation of the two-dimensional Laplacian, namely, the inequalities

$$\Delta u + \sum_{i=1}^{2} b_i(x) \frac{\partial u}{\partial x_i} \geqslant p(x) u^\sigma, \qquad x \in \mathbb{R}^2, \tag{4.8}$$

and

$$\Delta u + \sum_{i=1}^{2} b_i(x) \frac{\partial u}{\partial x_i} \geqslant p(x) e^u, \qquad x \in \mathbb{R}^2. \tag{4.9}$$

COROLLARY 4.2. *Suppose that there exists a continuous function* $b^*: [0, \infty) \to [0, \infty)$ *such that*

$$|b_i(x)| \leqslant b^*(|x|), \qquad x \in \mathbb{R}^2, \qquad i = 1, 2,$$
$$b^*(r) = O(r^{-1}) \qquad \text{as} \quad r \to \infty$$

and

$$\int_0^\infty b^*(s)\, ds < \infty.$$

(i) *If*

$$\liminf_{r \to \infty} (r^2 m(r))^{-1/(\sigma - 1)} \left(\int_{r_0}^r s \log\left(\frac{r}{s}\right) \cdot p_*(s)\, ds \right)^{-1} = 0, \tag{4.10}$$

then inequality (4.8) *has no positive entire solutions.*

(ii) *If*

$$\limsup_{r \to \infty} \log(r^2 m(r)) \left(\int_{r_0}^r s \log\left(\frac{r}{s}\right) \cdot p_*(s)\, ds \right)^{-1} \geqslant 0,$$

then inequality (4.9) *has no entire solutions.*

Proof. The proof is given only for (i). In this case we can take

$$T(r) = C_1, \qquad D(r) = 2 + C_2 r b^*(r), \qquad q(r) = p_*(r)$$

for some suitable constants $C_1, C_2 > 0$. Therefore, we see that

$$K(r) = r \exp\left(-2 \log \frac{r}{r_0} \right) \cdot \exp\left(-C_2 \int_{r_0}^r b^*(s)\, ds \right),$$

which implies that

$$C_3 r^{-1} \leqslant K(r) \leqslant C_4 r^{-1}, \qquad r \geqslant r_0$$

for some $C_3, C_4 > 0$. Hence, (4.10) implies (4.4), and the conclusion follows from Theorem 4.1 (i).

The example below shows that our nonexistence criteria are sharp in some sense.

EXAMPLE 4.2. Let $N \geqslant 3$. Consider the elliptic equation

$$Lu = \phi(x) u^\sigma, \qquad x \in \mathbb{R}^N, \tag{4.11}$$

where $\sigma > 1$ and a_{ij}, b_i, ϕ are continuous in $\mathbb{R}^N$. Suppose that the limits $\bar{a}_{ij} = \lim_{|x| \to \infty} a_{ij}(x)$ exist and the matrix $(\bar{a}_{ij})$ has positive eigenvalues. Suppose moreover that $b_i(x) = o(|x|^{-1})$ as $|x| \to \infty$, $1 \leqslant i \leqslant N$.

If $\phi(x) > 0$ in $\mathbb{R}^N$ and

$$\liminf_{|x| \to \infty} |x|^2 \phi(x) > 0,$$

then Eq. (4.11) has no positive entire solutions. This follows immediately from Example 4.1. On the other hand, if a_{ij}, b_i, ϕ are locally Hölder continuous in $\mathbb{R}^N$ and

$$\limsup_{|x| \to \infty} |x|^{2+\varepsilon} |\phi(x)| < \infty$$

for some $\varepsilon > 0$, then Eq. (4.11) has a positive entire solution $u(x)$ satisfying $u(x) \to c \in (0, \infty)$ as $|x| \to \infty$. This follows from a result of Friedman [2, Corollary 2]. According to Friedman's result there exist functions $w_i \in C^2(\mathbb{R}^N)$ such that

$$Lw_i = \frac{(-1)^i}{(1 + |x|)^{2+\varepsilon}}, \qquad x \in \mathbb{R}^N,$$

$$w_i(x) \to l \in (0, \infty) \qquad \text{as} \quad |x| \to \infty, \qquad i = 1, 2,$$

and

$$0 < w_2(x) \leqslant l \leqslant w_1(x), \qquad x \in \mathbb{R}^N.$$

Then, for sufficiently small $\alpha > 0$, the functions $\alpha w_1(x)$ and $\alpha w_2(x)$ can be shown to become a supersolution and a subsolution of Eq. (4.11), respectively. Hence, the standard supersolution and subsolution method (see, for example, [7, Theorem 2.10]) ensures the existence of a positive entire solution $u(x)$ of (4.11) squeezed between $\alpha w_1(x)$ and $\alpha w_2(x)$. Clearly, $u(x)$ tends to αl as $|x| \to \infty$.

Final Remark. All the results of this paper are still true even if the positivity condition of $p(x)$ is weakened to the condition that

$$p(x) \geqslant 0 \quad \text{in } \mathbb{R}^N, \quad \text{and} \quad p(x) > 0 \quad \text{for large } |x|.$$

The proofs are essentially the same.

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